# Game Theory

## Games of Incomplete Information

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# Introduction

- در بسیاری از برخوردهای دنیای واقعی طرفین اطلاعات کاملی از همدیگر ندارند.

مساله اطلاعات ناقص موجب بروز مشکلات بسیاری در "ایجاد بازارها" و رسیدن به حدود کارایی کامل می شود. به عنوان مثال:

  – کیفیت نیروی کاری که باید برای بنگاه کار کند مشخص نیست (فرآیند استخدام نیروی متخصص).
  – شرکت بیمه نمی تواند برای بیمه کردن افراد اطلاعات بدن آن ها را داشته باشد هرچند اگر هم بتواند ممکن است پزشک با یک معالجه نتواند همه بیماری ها را شناسایی کند در حالی که خود بیمار ممکن است بداند.
  – رئیسان نمی توانند دائماً از کیفیت کار کارمندان خود مطمئن شوند بنابراین آنها ممکن است کار نکنند و وقت خود را به بطالت و یا انجام کارهای شخصی بگذرانند.
  – مدیران دولتی ممکن است به خودشان پاداش بدهند چراکه انگیزه ای برای بهبود کارایی ندارند و برای دولت نیز بسیار پرهزینه است که آن ها را کنترل کند. (البته انگیزه دولت نیز مخدوش است)
  – ازدواج یک نوع معامله با اطلاعات ناقص است چرا که شما نمی دانید طرف مقابل با چه انگیزه ای به شما علاقه نشان می دهد!
  – هنگام خرید یک کالا که اطلاعی از فرآیند و هزینه تولید آن ندارید معمولاً به قیمت توجه می کنید. آیا هر کالای گرانتری بهتر نیز هست؟
  – چگونه پزشک معالج خود را انتخاب می کنید؟ (اطلاع پزشک بیشتر از شماست)
  – مولفه ای در کیفیت کار یک پزشک، یک محقق، یک معلم و یا یک متخصص در هر زمینه ای وجود دارد که قابل مشاهده نیست.

# Introduction

- In the PGs section we relaxed the assumption that “each player possesses all the relevant information about the other players’ payoffs”.

- In this section we will further relax the assumption of complete information and move on to games of *incomplete information* where players may not know others’ payoff functions

- Incomplete information games is a rapidly growing area in game theory

  In 2001, the Nobel prize was awarded to the three pioneers of information economics:

  - ✓ Product markets: sellers know the quality of their good but the buyers don't (George Akerlof, 1970)
  - ✓ Labour markets: workers know their abilities but the employers don't (Michael Spence, 1973)
  - ✓ Credit markets: entrepreneurs know the risk of their projects but the investors and banks don't (Joseph Stiglitz, 1976)

# Introduction

What is new in the analysis of incomplete information?

- In the 1960s, the micro paradigm of general equilibrium theory was dominant: all actors are price takers and there is perfect information about all relevant parameters
  - With private information, general equilibrium theory loses much relevance as the information asymmetry can be used strategically

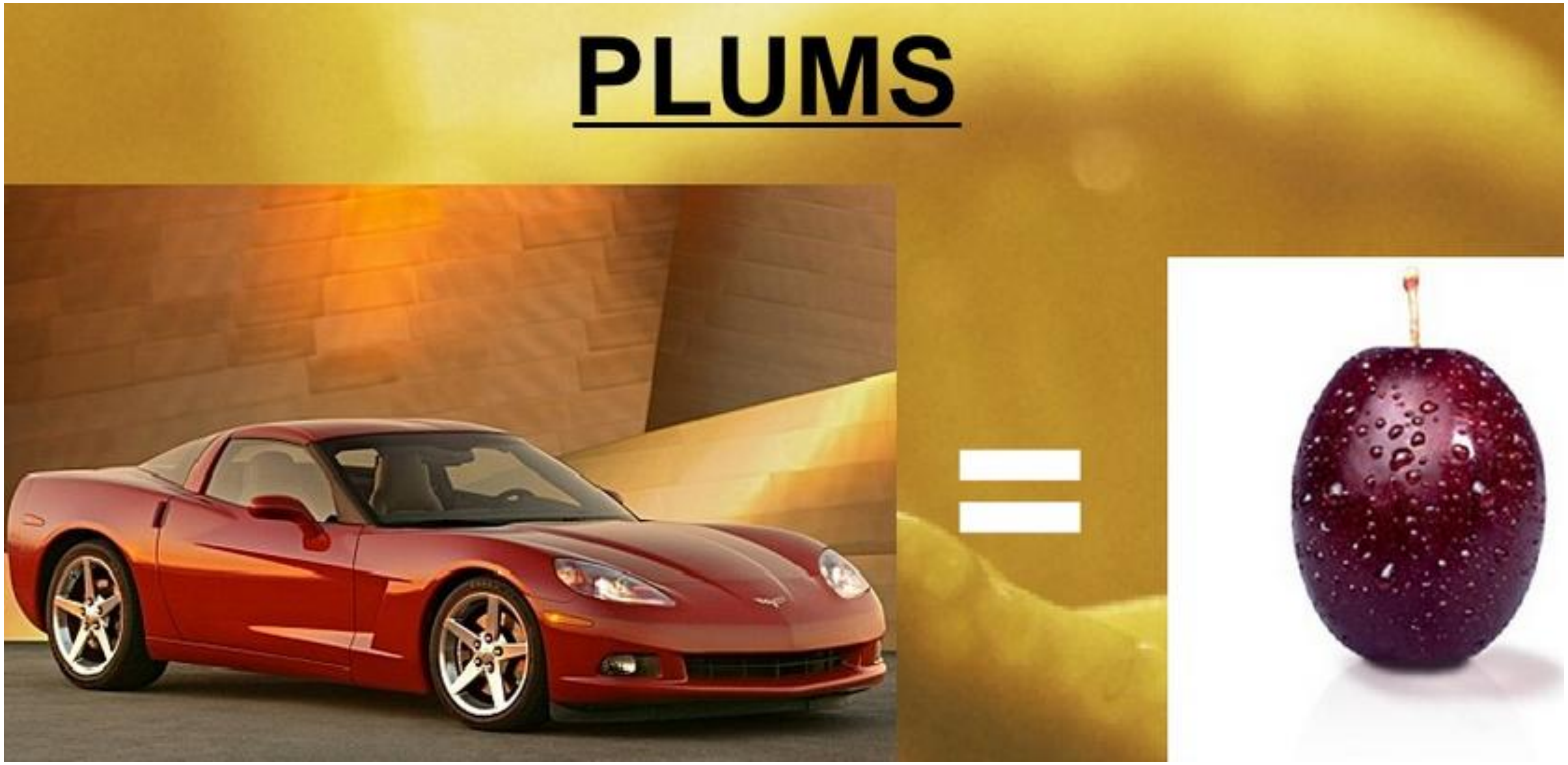
- Akerlof was the first one who in his 1970's paper suggested a model with asymmetric information. His paper is about quality and uncertainty.

- Akerlof (1970) in an example of *"Automobiles market"* called *"market for lemons"* provide a reasoning why "Business in underdeveloped countries is difficult"; in particular, a structure is given for determining the economic costs of dishonesty.

# Market for Lemons


- A lemon is typically a euphemism for a bad quality car
- Try to avoid choosing a lemon when you use the second hand car market!

# Market for Lemons




- A plum is used as a euphemism for a good quality car
- Obviously every buyer would desire a ‘Plum’

# Market for Lemons

- In an Automobiles market there are two types of cars “Plum” and “Lemon”.
- Buyers willingness to pay and sellers perceived values for each type are as follows:

| Types | Plums | Lemons |
|---|---|---|
| *Probability* | *1/3* | *2/3* |
| Buyers’ WTP | 3000 | 2000 |
| Sellers’ real valuation | 2400 | 1200 |

$\Rightarrow$ Expected value of a car = 1/3(3000)+2/3(2000) = 2333.3

- Plums' offer will not be accepted (‘cause 2400 > 2333.3)
- Lemons' offer will be accepted (‘cause 1200 < 2333.3)
- There is no market for Plums. This normally happens in developing countries.

# Market for Lemons

- Now change the probabilities in favor of having more Plums which resembles the markets in developed countries.
- Buyers willingness to pay and sellers perceived values for each type are as follows:

| **Types** | **Plums** | **Lemons** |
|---|---|---|
| *Probability* | *2/3* | *1/3* |
| Buyers' WTP | 3000 | 2000 |
| Sellers' real valuation | 2400 | 1200 |

$\Rightarrow$ Expected value of a car = 1/3(2000)+2/3(3000) = 2666.6

- ➢ Plums' offer will be accepted ('cause 2400 < 2666.6)
- ➢ Lemons' offer will be accepted ('cause 1200 < 2666.6)
- ➢ There is market for Plums. This normally happens in developed countries.

# Agenda

1. Static Games
    1.1. Harsanyi's notion of imperfect information
    1.2. Bayesian games
    1.3. Cournot duopoly with asymmetric information
    1.4. A first-price sealed-bid auction

2. Dynamic Games
    2.1 Introduction
    2.2 Perfect Bayesian equilibrium
    2.3 Equilibrium refinements

# Static games

- We will first consider static versions of incomplete information games
  - E.g. Firms do not know their opponents' cost (oligopoly)
  - E.g. Type of preferences is unknown (auctions)
  - E.g. Quality of player i is unknown to player j (education)
- Consider a modified version of the PD game with uncertainty about prisoner 2's payoff. Here, Prisoner 1 dislikes to defect if prisoner 2 does not defect (DC is not dominated for P1); Prisoner 1 could be called a "conditional cooperator"

# An introductory example: PD

Source: MWG, p. 254

| Prisoner 1 \ Prisoner 2 | DC | C |
|---|---|---|
| DC | 0, −2 | −10, −1 |
| C | −1, −10 | −5, −5 |

Case 1:
Confess is a dominant strategy for Prisoner 2
→{C, C} is the Nash equil.
Let case 1 occur with probability μ

| Prisoner 1 \ Prisoner 2 | DC | C |
|---|---|---|
| DC | 0, −2 | −10, −7 |
| C | −1, −10 | −5, −11 |

Case 2:
DC is a dominant strategy for Prisoner 2
→{DC, DC} is the Nash equil.
Let case 2 occur with probability 1 – μ

# Harsanyi's notion of imperfect information

Harsanyi (1967) introduced the idea that a player called "Nature" or "Chance" determines player 2's *type*

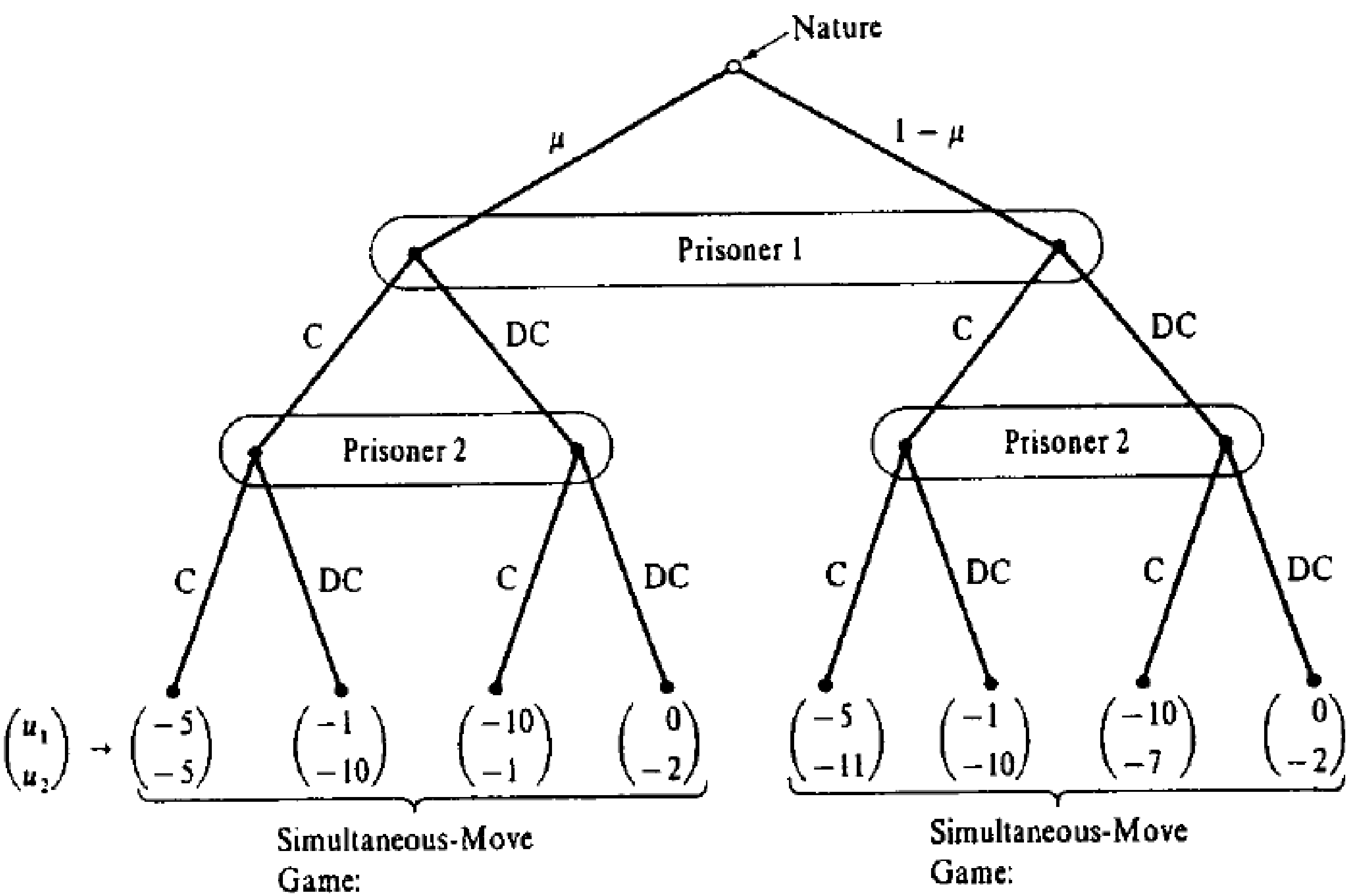



**Figure 8.E.1**
The DA's Brother game with incomplete information.

# Harsanyi's notion of imperfect information

- This is a transformation of the *incomplete information* game into one with complete but *imperfect information*
- Remember
  - Imperfect Information: For some move, some player does not know the complete history of the game (does not know where in the game tree she is)

- Transformation
  1. We add player "Nature" → Harsanyi transformation
  2. Different payoff functions → realization of type $\theta i$
  3. Type $\theta i$ is known to player i but not to player j
  4. Player i has beliefs $p(\theta\text{-}i \mid \theta i)$: this is the conditional probability of player i on his opponents' type given $\theta i$ (although, in many applications, $p(\theta\text{-}i)$ does actually not depend on $\theta i$)

# Bayesian games

The normal form of a game with imperfect information (*Bayesian game*) consists of

- A number of players
- Action spaces $A_i$
- Type spaces $\Theta_i$ with $\theta_i$ *in* $\Theta_i$
- Probability distribution $p(\theta_{-i} / \theta_i)$ coming from the joint probability distribution $p(\theta_1, \ldots, \theta_I)$ which is a common knowledge
- Payoff functions $u_i (s_i, s_{-i}, \theta_i)$

Although in the simultaneous-move games we consider players simply choose an action $a_i$ *in* $A_i$, we need to specify a strategy $s_i (\theta_i)$ for each player which is an action for each possible type of all players

# Bayesian Nash equilibrium

**Definition:** A *Bayesian Nash Equilibrium* in a Bayesian game is a combination of type-dependent strategies $(s_1^*(\theta_1), \ldots, s_I^*(\theta_I))$ such that every player maximizes her expected utility, given $\theta_i$ and the strategies of all other players through choice of $s_i^*(\theta_i)$

**Formally**

$$s_i^*(\theta_i) \in \arg\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i) u_i \left(s_1^*(\theta_1), \ldots, a_i, \ldots, s_I^*(\theta_I)\right)$$

for each $\theta_i$ *in* $\Theta_i$.

- In MWG the notation is different:

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \mid \bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i', s_{-i}(\theta_{-i}), \bar{\theta}_i) \mid \bar{\theta}_i]$$

- Remember:

$$p(\theta_{-i} \mid \theta_i) = \frac{p(\theta)}{p(\theta_i)} = \frac{p(\theta)}{\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}, \theta_i)}$$

# Bayesian Nash equilibrium: PD

- Player 2 will always confess in case 1, but will never confess in case 2

- Payoff for player 1:

$$DC: \quad -10\mu + 0(1-\mu)$$

$$C: \quad -5\mu - 1(1-\mu)$$

$$-10\mu + 0(1-\mu) > -5\mu - 1(1-\mu) \equiv \mu < 1/6$$

→ Player 1 chooses DC if $\mu < 1/6$ and $C$ if $\mu > 1/6$ (indifferent if $\mu = 1/6$)

- (Note that we usually denote p for beliefs but here we stick to $\mu$ as in the MWG example)

# Trembling-hand perfection

Following Selten (1975), for any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, we can define a *perturbed* game $\Gamma_\varepsilon = [I, \{\Delta_\varepsilon(S_i)\}, \{u_i(\cdot)\}]$ by choosing for each player $i$ and strategy $s_i \in S_i$ a number $\varepsilon_i(s_i) \in (0, 1)$, with $\sum_{s_i \in S_i} \varepsilon_i(s_i) < 1$, and then defining player $i$'s perturbed strategy set to be

$$\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \sum \sigma_i(s_i) = 1\}.$$

**Definition 8.F.1:** A Nash equilibrium $\sigma$ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (*normal form*) *trembling-hand perfect* if there is *some* sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ that converges to $\Gamma_N$ [in the sense that $\lim_{k\to\infty} \varepsilon_i^k(s_i) = 0$ for all $i$ and $s_i \in S_i$], for which there is *some* associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to $\sigma$ (i.e., such that $\lim_{k\to\infty} \sigma^k = \sigma$).

- Having the possibility of mistakes we may define a ***perturbed game*** by choosing for each player I and strategy $s_i$ *in* $S_i$ a number $0 \leq \varepsilon_i \leq 1$such that $\sum_i \varepsilon_i = 1$.
- By accruing this sequence of mistakes to a game that converges to the main game, an equilibrium σ is ***trembling-hand perfect*** if the corresponding sequence of NE $\sigma^\varepsilon$ converges to σ.

# Trembling-hand perfection

| | $L$ ($\varepsilon$) | $R$ ($1 - \varepsilon$) |
|---|---|---|
| $U$ | 1, 1 | 0, −3 |
| $D$ | −3, 0 | 0, 0 |

**Figure 8.F.1**
($D$, $R$) is a Nash equilibrium involving play of weakly dominated strategies.

For trembling hand perfection, take small non-intentional “trembles" of the other player into account such that the other player plays the equilibrium with prob. $1 - \varepsilon$ and the other strategy with prob. $\varepsilon$

- U → $\varepsilon(1) + 0(1 - \varepsilon) = \varepsilon$
- D → $\varepsilon(-3) + 0(1 - \varepsilon) = -3\varepsilon$

⇨ For $\varepsilon \to 0$; deviating to U pays and thus (D, R) is not a THP Nash equilibrium

# Cournot duopoly with asymmetric information

- Consider a Cournot duopoly with linear demand:

$$\text{price} = a - q_1 - q_2$$

- Now suppose that firm 1's cost is known to be c but firm 2's cost is either $c_H$ or $c_L$, where $c_H > c_L$ and the firms know

$$\text{prob}(c_2 = c_H) = \rho$$

- Firm 2 has two possible payoff functions:

$$\pi_2(q_1, q_2, c_H) = (a - q_1 - q_2 - c_H)q_2$$

$$\pi_2(q_1, q_2, c_L) = (a - q_1 - q_2 - c_L)q_2,$$

- whereas firm 1 has only got one:

$$\pi_1(q_1, q2, c) = (a - q_1 - q_2 - c)q_1$$

# Cournot duopoly with asymmetric information

Formal definition of this Cournot game

- Players: Firm 1 and Firm 2
- Action spaces: $A_1 = A_2 = [0, \infty)$
- Type spaces: $\Theta_1 = \{c\}$, $\Theta_2 = \{c_H, c_L\}$
- Beliefs: $prob(c_2 = c_H) = \rho$, $prob(c_2 = c_L) = 1 - \rho$
- Payoffs:

$$\pi_1(q_1, q_2, c) = (a - q_1 - q_2 - c)q_1$$

$$\pi_2(q_1, q_2, c_H) = (a - q_1 - q_2 - c_H)q_2$$

$$\pi_2(q_1, q_2, c_L) = (a - q_1 - q_2 - c_L)q_2$$

# Cournot duopoly with asymmetric information

- If a firm knows her payoff function, we say that she knows her type. Similarly, if a player is uncertain about the payoff function of others, we say that there is uncertainty about the type of the other player(s)

- Intuitively, firm 2's output decision will depend on the cost parameter. Firm 1 would prefer to know exactly against which type she is playing but she only knows $\theta$ and $\rho$

- We will solve this game by looking for the Bayesian Nash equilibrium

# Cournot duopoly with asymmetric information

- We begin with the two types of firm 2. They will choose:

$$\underset{q_2}{Max}\ \left(a - q_1^* - q_2 - c_H\right)q_2 \xrightarrow{Gives} q_2^*(c_H)$$

$$\underset{q_2}{Max}\ \left(a - q_1^* - q_2 - c_L\right)q_2 \xrightarrow{Gives} q_2^*(c_L)$$

- Therefore the firm 1 will solve:

$$\underset{q_1}{Max}\ \rho\left(a - q_1 - q_2^*(c_H) - c\right)q_1 + (1-\rho)\left(a - q_1 - q_2^*(c_L) - c\right)q_1$$

to find $q_1^*$.

- The three first-order conditions read:

# Cournot duopoly with asymmetric information

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2}$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}$$

$$q_1^* = \frac{\rho\left(a - q_2^*(c_H) - c\right) + (1-\rho)\left(a - q_2^*(c_L) - c\right)}{2} = \frac{a - c - E_\theta(q_2^*)}{2}$$

- The (Nash) solution to this system of equations is:

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1-\rho}{6}(c_H - c_L) = \frac{a - 2c_H + c}{3} + \frac{c_H - E_\theta(c)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_H + c}{3} - \frac{\rho}{6}(c_H - c_L) = \frac{a - 2c_L + c}{3} - \frac{E_\theta(c) - c_L}{6}$$

$$q_1^* = \frac{a - 2c + \rho c_H + (1-\rho)c_L}{3} = \frac{a - 2c + E_\theta(c)}{3}$$

$q_1^*$ generally does not have this form. E.g. When $TC = c_i q_i^2$

# Cournot duopoly with asymmetric information

We make several observations here

1. All three equilibria output choices depend on all three cost parameters $c$, $c_H$ *and* $c_L$.
2. The **first best complete information equilibria** of the Cournot duopoly with asymmetric cost is

$$q_i^* = (a - 2c_i + c_j)/3$$

   – Firm 2 with $c_2 = c_H$ produces more than this complete information benchmark
   – Firm 2 with $c_2 = c_L$ produces less than the complete information benchmark
   – For $\rho \rightarrow 1$ and $\rho \rightarrow 0$, we approach the complete information benchmark for $c_H$ and $c_L$, respectively

# A first-price sealed-bid auction

We will now analyze a private-value auction. The format is the first-price sealed-bid auction (see Gibbons, 1992)

1. There are two bidders bidding for a single object
2. Their valuations are $v_1$ and $v_2$
3. They simultaneously submit their sealed bids
4. The bidder with the highest evaluation wins and pays the price she bid
5. The other bidder pays nothing

In the case of a tie (equal bids), the winner is determined by the toss of a coin.

**And** *bidders are risk neutral.*

# A first-price sealed-bid auction

Players: bidder 1 and bidder 2

Action spaces: $A_1 = A_2 = [0, \infty)$, actions are $b_1$ and $b_2$

Type spaces: $v_1 \in \Theta_1 = [0,\ 1],\ v_2 \in \Theta 2 = [0,\ 1]$

Beliefs: Player i believes that $v_j$ is distributed *uniformly* on *[0, 1]*

Payoffs:

$$u_i(b_1, b_2, v_i) = \begin{cases} v_i - b_i & if \quad b_i > b_j \\ (v_i - b_i)/2 & if \quad b_i = b_j \\ 0 & o.w. \end{cases}$$

# A first-price sealed-bid auction

Note that we have infinitely many types for both players here, and we have to specify an action for all of them

We do this by analyzing bidding functions, $b_i(v_i)$, specifying what each type (each valuation) would bid. However, we may assume that player j adopts the strategy b(.), and assume that b(.) is *strictly increasing* and *differentiable*:

$$\underset{b_i}{Max}\,(v_i - b_i)\cdot \Pr\left(b_i > b\left(v_j\right)\right)$$

Since $v_j$ is uniformly distributed on *[0,1]*:

$$\Pr\left(b_i > b\left(v_j\right)\right) = \Pr\left(v_j < b^{-1}(b_i)\right) = b^{-1}(b_i) = v_i$$

The FOC for player i’s optimization problem is therefore:

$$-b^{-1}(b_i) + (v_i - b_i)\frac{1}{b'(v_i)} = 0$$

# A first-price sealed-bid auction

Solving the differential equation:

$$b'(v_i)v_i + b_i(v_i) = v_i$$

Find:

$$b_i(v_i)v_i = \frac{1}{2}v_i^2 + k$$

Observe that $b(v_i) \leq v_i$; therefore b(0)=0. This means that k=0 and finally:

$$b_i(v_i) = \frac{1}{2}v_i$$

That is, each bidder bids half her valuation (see Gibbons 1992, section 3.2B)

# A first-price sealed-bid auction: Experiment

Let us consider one more "treasure" and an "intuitive contradiction" by Goeree and Holt (2001)

✓ *In auction experiments with private values, people are known to overbid relative to the Nash equilibrium.*

- Goeree and Holt explore this issue further:
  - ➢ They have two auction experiments:
  - ➢ In the first, the possible valuations are $0, $2, and $5
  - ➢ In the second, they are $0, $3 and $6
  - ➢ In both experiments, all three valuations are equally likely
  - ⇒ The Nash equilibrium bids are $0, $1, and $2 for *both* treatments, as can be seen in the following tables

# A first-price sealed-bid auction: Experiment

| | bid = 0 | bid = 1 | bid = 2 | bid = 3 | bid = 4 | bid = 5 |
|---|---|---|---|---|---|---|
| value = $0 | 0* | -.5 | -1.66 | - 3 | - 4 | - 5 |
| value = $2 | .33 | .5* | 0 | -1 | -2 | - 3 |
| value = $5 | .83 | 2 | 2.5* | 2 | 1 | 0 |

| | bid = 0 | bid = 1 | bid = 2 | bid = 3 | bid = 4 | bid = 5 |
|---|---|---|---|---|---|---|
| value = $0 | 0* | -.5 | -1.66 | - 3 | - 4 | - 5 |
| value = $3 | .5 | 1* | .83 | 0 | -1 | - 2 |
| value = $6 | 1 | 2.5 | 3.33* | 3 | 2 | 1 |

$0, $2, $5 treatment (top) and $0, $3, and $6 treatment in Goeree and Holt (2001). The entries denote the expected payoffs from bidding $x, given value $y. Nash equilibrium bids are marked with *

# A first-price sealed-bid auction: Experiment

- ❖ The results of experiment:
- ✓ There is *overbidding* in the $0, $2, $5 experiment, too, BUT much more so in the $0, $3, $6 treatment!

- ✓ In the $0, $2, $5 treatment, average bids were:
  - ✓ 0, 1.04, and 2.64, respectively, with 80% equilibrium bids

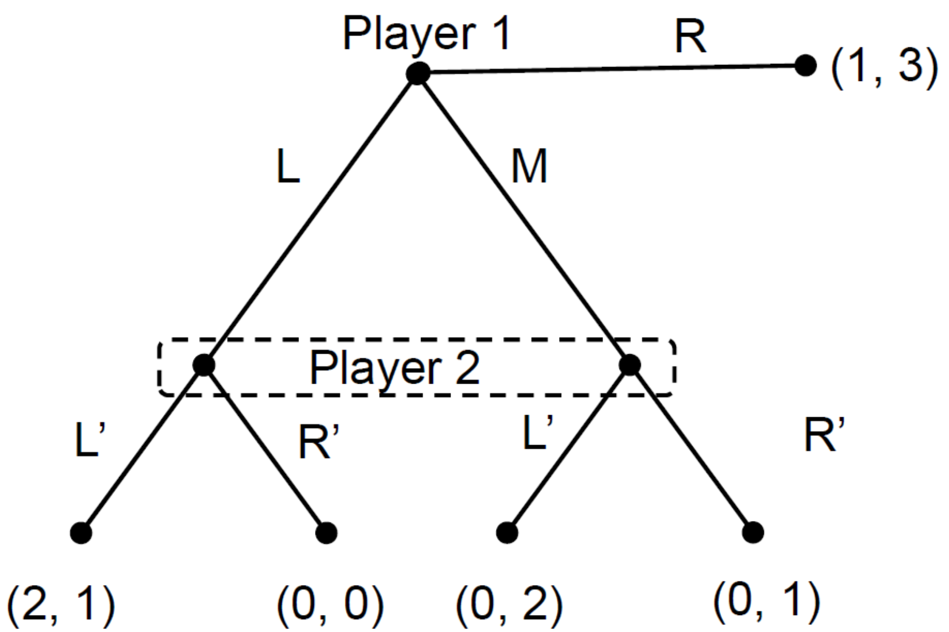
- ✓ In the $0, $3, $6 treatment, average bids were:
  - ✓ 0, 1.82, and 3.40, respectively with 50% equilibrium bids

- ✓ While bids in the $0, $2, $5 treatment are close to Nash (treasure), they are quite remote from Nash in the second

- ✓ What may account for the differences?

# 2- Dynamic games of incomplete info

- The treatment of incomplete information is similar as in the static games but more involved as we will be dealing with information revelation and disclosure across multiple periods or stages
- Consider this example: player 1 can end the game by choosing R. If R is not chosen, Player 2 does not know whether L or M was selected (see Gibbons, 1992, p. 176)




The two Nash equilibria are (L, L') and (R, R') but we note that (R, R') is based on a non-credible threat

# 2- Dynamic games of incomplete info

- In games with complete information, we used subgame perfection to get rid of the non-credible equilibrium.

- Problem: applying subgame perfection here does not help as there are no subgames

- Recall that subgames begin at singleton decision nodes; here, the only subgame is the entire game. As a result, *subgame perfection* is trivially satisfied

$\Rightarrow$ We need a new equilibrium concept here

- For dynamic games of incomplete information, we will apply *Perfect Bayesian Equilibrium (PBE)*

# PBE

**Definition:** *A Perfect Bayesian Equilibrium consists of strategies and beliefs satisfying the following requirements:*

1. *At each information set, the player to move must hold a **belief** about which node has been reached*

2. *Given their beliefs, players' strategies must be **sequentially rational** (that is, must be optimal given the beliefs and rivals' strategies)*

3. *Beliefs are determined by **Bayes' rule** at information sets on and off the equilibrium path (where off the equilibrium path means that this information set occurs with probability zero)*

# PBE

**Definition 9.C.1:** A *system of beliefs* $\mu$ in extensive form game $\Gamma_E$ is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node $x$ in $\Gamma_E$ such that

$$\sum_{x \in H} \mu(x) = 1$$

for all information sets $H$.

**Definition 9.C.2:** A strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ in extensive form game $\Gamma_E$ is *sequentially rational at information set H given a system of beliefs* $\mu$ if, denoting by $\iota(H)$ the player who moves at information set $H$, we have

$$E[u_{\iota(H)} | H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} | H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile $\sigma$ satisfies this condition for *all* information sets $H$, then we say that $\sigma$ is *sequentially rational given belief system* $\mu$.

# PBE

**Definition 9.C.3:** A profile of strategies and system of beliefs $(\sigma, \mu)$ is a *weak perfect Bayesian equilibrium* (weak PBE) in extensive form game $\Gamma_E$ if it has the following properties:

(i) The strategy profile $\sigma$ is sequentially rational given belief system $\mu$.

(ii) The system of beliefs $\mu$ is derived from strategy profile $\sigma$ through Bayes' rule whenever possible. That is, for any information set $H$ such that $\text{Prob}(H \mid \sigma) > 0$ (read as "the probability of reaching information set $H$ is positive under strategies $\sigma$"), we must have

$$\mu(x) = \frac{\text{Prob}(x \mid \sigma)}{\text{Prob}(H \mid \sigma)} \quad \text{for all } x \in H.$$

**Proposition 9.C.1:** A strategy profile $\sigma$ is a Nash equilibrium of extensive form game $\Gamma_E$ if and only if there exists a system of beliefs $\mu$ such that

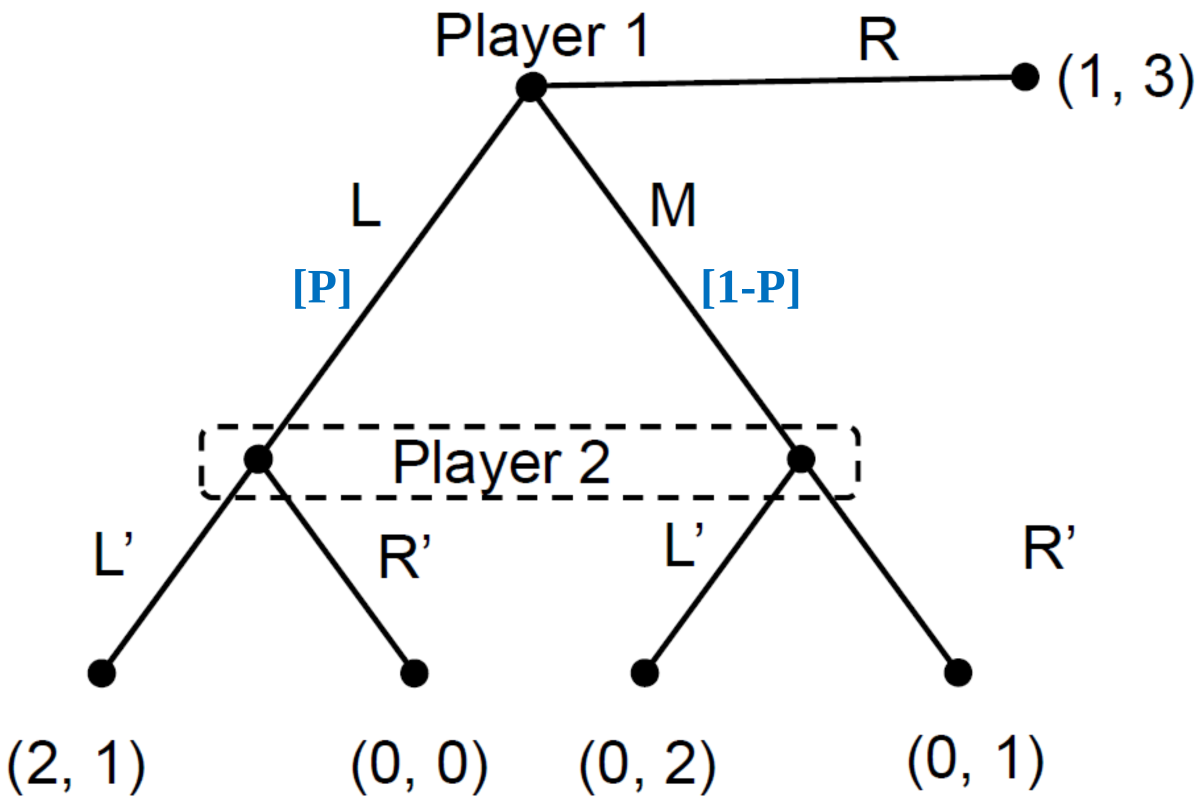
(i) The strategy profile $\sigma$ is sequentially rational given belief system $\mu$ *at all information sets $H$ such that Prob $(H \mid \sigma) > 0$.*

(ii) The system of beliefs $\mu$ is derived from strategy profile $\sigma$ through Bayes' rule whenever possible.

***PBE => NE, but not vice versa***

# PBE

In the introductory example, part 1 of the definition amounts to assigning beliefs for Player 2 whether L (*prob.=p)* or M *(prob.= 1 - p)* was chosen by Player 1:



# PBE

*Part 2. of the definition (sequential rationality)* requires that players maximize expected payoffs given the beliefs:

- Player 2's expected payoff from choosing R' is

$$p \cdot 0 + (1-p) \cdot 1 = 1 - p$$

- Player 2's expected payoff from choosing L' is

$$p \cdot 1 + (1-p) \cdot 2 = 2 - p$$

Thus sequential rationality prevents player 2 from choosing R', regardless of the belief *p.*

⇒ *This eliminates* the implausible equilibrium (R, R')

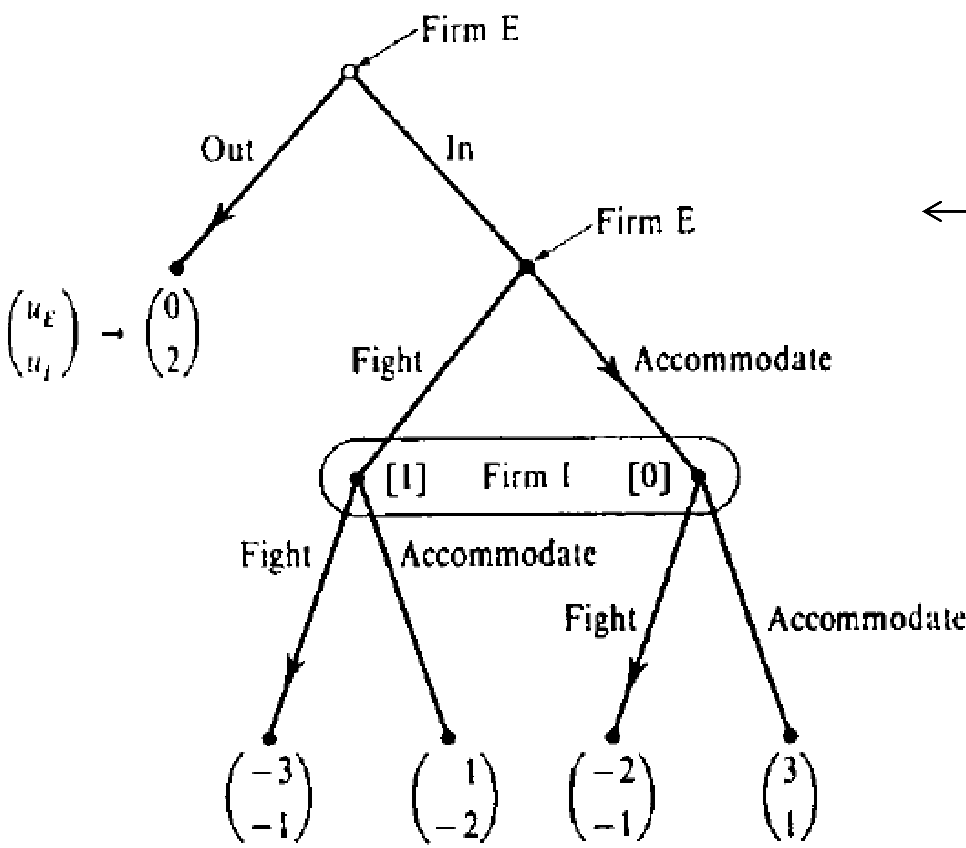
⇒ <u>This strategy-beliefs pair is the unique PBE in this game</u>

# Discussion

- The concept of PBE puts very minimal requirements on the consistency of equilibriums → It's too positive

✓ *No restrictions at all are placed on out-of-equilibrium beliefs*

- Example:




✓ There are multiple PBEs

❖ One of them is shown here being based on the out-of-equilibrium belief for firm I that assign probability 1 to firm E having played "fight".

❑ But these strategies are not even SPNE!

# Equilibrium refinements

- An odd point in the analysis of the PBE was that we were free to fix beliefs for an out-of-equilibrium event
- This multitude of equilibria has been called an “embarrassment of riches” for game theory
- There seems to be no predictive power emerging from these models—anything goes!
- Game theorists in 1990s therefore spent a huge amount of effort developing criteria based on which we can select among the set of perfect Bayesian equilibria
- This is the literature on *equilibrium refinements*
- E.g. *Intuitive Criterion* in *signaling* is a type of *equil. refinement*

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